Decomposing R^2 with the Owen value

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Abstract

We provide an axiomatization-based justification for applying the Owen value to decompose R^2 in OLS models if prior knowledge can be used to form groups of regressor variables. The assumptions made by the axioms are not only plausible with respect to the variables but also clarify the meaning of the exogenous grouping of variables.

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1 Introduction

The Shapley (1953) value can be used to decompose goodness-of-fit measures of regression models—such as R^2—into contributions of individual regressor variables (Chevan and Sutherland, 1991; Johnson and LeBreton, 2004). In many practical applications there are good reasons to think in terms of groups of regressors if they represent similar explanatory concepts, e.g. a polynomial in labor market experience, dummy variables created from a categorical variable, or savings and investment rates in a growth model. In such situations, it is desirable to decompose the model's R^2 into contributions of groups and—if of interest—to allocate each group's share to its constituent variables. The Owen (1977) value provides an obvious candidate for a decomposition procedure incorporating exogenously defined groups (Shorrocks, 1999).

An obstacle to wide-spread application of R^2 decomposition in applied work may have been confusion about the implied assumptions. We argue that the Owen value R^2 decomposition arises under very mild and plausible assumptions. It is also attractive in many practical settings inasmuch as it usually allows for models with more regressor variables if not all groups contain only a single variable. If all groups are singletons, though, the Owen value coincides with the Shapley value. Thus, the proposed procedure can be regarded as a generalization of the Shapley value R^2 decomposition. After a presentation of the method and its theoretical underpinnings, we provide an illustrative example for a wage regression with German data.

2 Method

Let $K = \{1, ..., j, ..., k\}$ denote the set of explanatory variables or—in the language of game theory—the set of players. With the interpretation mentioned in the introduction, we may group the variables and obtain the partition $\mathcal{G} = \{G_1, ..., G_\ell, ..., G_\gamma\}$. Estimation of the full OLS model

$$y = \beta_0 + \underbrace{\beta_1 x_1 + \ldots}_{G_1} + \ldots + \underbrace{\ldots + \beta_j x_j + \ldots}_{G_\ell} + \ldots + \underbrace{\ldots + \beta_k x_k}_{G_\gamma} + \varepsilon$$

gives the $R^2(K)$ to be distributed. To this end, running a regression for every combination of variables $T \subseteq K$ and the resulting $R^2(T)$ will be considered. Since a regression without any variable but the intercept yields zero explained variance, we face a so-called cooperative game with transferable utility and a coalition structure (CS-game) with player set K, coalition function $R^2 : \mathbf{2}^K \longrightarrow \mathbb{R}$, $R^2(\emptyset) = 0$, where $\mathbf{2}^K$ denotes the power set of K, and coalition structure $\mathcal{G}$.

A solution concept ϕ prescribes for any given $\left(K, R^2, \mathcal{G}\right)$ the variables' payoffs, i.e. it assigns to every CS-game a vector $\phi\left(K, R^2, \mathcal{G}\right) \in \mathbb{R}^K$ where $\phi_j\left(K, R^2, \mathcal{G}\right)$ is interpreted as variable j's payoffs. Note, a solution ϕ does not attribute a value to the group; however, summing up the variables' payoffs in a group, $\sum_{j \in G_\ell} \phi_j\left(K, R^2, \mathcal{G}\right)$, can describe the group's payoff.

A CS-game $\left(K, R^2, \mathcal{G}\right)$ induces a game played between the groups. It is called the external game and is denoted by $\left(\mathcal{G}, R^2_{\mathcal{G}}\right)$, where the coalition function assigns to every set of groups $\Upsilon \subseteq \mathcal{G}$ its worth via $R^2_{\mathcal{G}}(\Upsilon) = R^2\left(\cup_{G_\ell \in \Upsilon} G_\ell\right)$, i.e. $R^2_{\mathcal{G}}(\Upsilon)$ is the R^2 obtained from estimation of the OLS model

$$y = \beta_0 + \sum_{G_\ell \in \Upsilon} \sum_{j \in G_\ell} \beta_j x_j + \varepsilon.$$

2.1 Owen value R^2 decomposition

The Owen value is well-suited for a decomposition of $R^2(K)$ of the full model because of its characteristic properties and because it is the consistent extension of the Shapley value.

2.1.1 Axiomatic argument for the Owen value R^2 decomposition

Allocation of R^2 to (groups of) variables has to involve certain assumptions. In the following, we impose four assumptions that we regard as plausible.

Efficiency: The full model R^2 is decomposed among the variables, i.e. $\sum_{j \in K} \phi_j\left(K, R^2, \mathcal{G}\right) = R^2(K)$.

Symmetry within groups: If j' and j'' are substitutes according to their explanatory power and belong to the same group, they obtain equal value, i.e.,

$$R^2\left(T \cup \{j'\}\right) = R^2\left(T \cup \{j''\}\right) \text{ for all } T \subseteq K \backslash \{j', j''\}$$

and $j', j'' \in G_\ell$ implies $\varphi_{j'}\left(K, R^2, \mathcal{G}\right) = \varphi_{j''}\left(K, R^2, \mathcal{G}\right)$.

Note that this symmetry condition ensures that symmetric players cannot be distinguished on the grounds of their group affiliation.

Symmetry between groups: If $G_{\ell'}$ and $G_{\ell''}$ are substitutes according to their explanatory power, they obtain equal valuation, i.e.,

$$R^2_{\mathcal{G}}\left(\Upsilon \cup \{G_{\ell'}\}\right) = R^2_{\mathcal{G}}\left(\Upsilon \cup \{G_{\ell''}\}\right) \text{ for all } \Upsilon \subseteq K \backslash \{G_{\ell'}, G_{\ell''}\}$$

implies $\sum_{j' \in G'} \varphi_{j'}\left(K, R^2, \mathcal{G}\right) = \sum_{j'' \in G''} \varphi_{j''}\left(K, R^2, \mathcal{G}\right)$.

That is to say that one abstracts from the number of variables in a group and raises groups on an equal level. If, e.g., a long list of grouped dummy variables has the same predictive power as a single variable group, the dummies in total receive the same share as the single variable.

Monotonicity: A change of the observations that leads to new coefficients of determination $\tilde{R}^2$ such that variable j exhibits higher marginal contributions, must not decrease the explanatory value attributed to variable j, i.e.,

$$\tilde{R}^2\left(T \cup \{j\}\right) - \tilde{R}^2\left(T \cup \{j\}\right) \geq R^2\left(T \cup \{j\}\right) - R^2\left(T \cup \{j\}\right) \text{ for all } T \subseteq K \backslash \{j\}$$

implies $\phi_j\left(K, \tilde{R}^2, \mathcal{G}\right) \geq \phi_j\left(K, R^2, \mathcal{G}\right)$.

For practical purposes, this condition introduces a kind of merit principle: variables with higher marginal contributions receive a higher payoff.

Theorem 1 (Khmelnitskaya and Yanovskaya, 2007) *The Owen value satisfies efficiency, symmetry within groups, symmetry between groups, and monotonicity.*[1]

The theorem employs axioms that refer to the abstract groups only for symmetry conditions. In contrast, the powerful monotonicity condition refers directly to the variables.

2.1.2 Relation to the axioms characterizing the Shapley value

The Shapley value is promoted in the literature to be applied for decomposing the R^2 if the variables are ungrouped. Indeed, the model and the characterization above

[1] In the more general class of CS-games, the Owen value is actually the only value that satisfies these conditions.

generalize the Shapley value R^2 decomposition in the sense that if $\mathcal{G}$ is the atomistic partition or consists of only one group—i.e. if the variables cannot be distinguished with respect to $\mathcal{G}$—, the framework above captures the ungrouped situation. Moreover, in this case the Owen value R^2 decomposition equals the Shapley value R^2 decomposition. To see this, adapt the efficiency and monotonicity properties above to the case where no coalition structure $\mathcal{G}$ exists. Further, the symmetry conditions above must be replaced by the following property, and the previous theorem unfolds as a generalization of the well-known result of Young (1985).

Symmetry: If variables j' and j'' are substitutes according to their explanatory power, they obtain equal value, i.e.,

$$R^2\left(T \cup \{j'\}\right) = R^2\left(T \cup \{j''\}\right) \text{ for all } T \subseteq K \backslash \{j', j''\}$$

implies $\phi_{j'}\left(K, R^2\right) = \phi_{j''}\left(K, R^2\right)$.

Corollary 2 *The Shapley value satisfies efficiency, symmetry, and monotonicity.*

2.2 Formulas for the Shapley value and the Owen value

In addition to the theorem above, it is useful to develop an intuition about what drives the solutions. We therefore provide both, intuitive formulas referring to marginal contributions and those used for computation.

2.2.1 Calculation of the Shapley value

We want to consider permutations of K, i.e. bijective mappings $\pi : K \to \{1, ..., |K|\}$ where $\pi(j)$ is interpreted as the position of player j in π. The set of all $k!$ permutations on K is referred to by $\Pi(K)$. For a given π, the predecessors of player j form a set denoted by $P_j^\pi = \{j' \mid \pi(j') < \pi(j)\}$. A player j's marginal contribution (if players are ordered according to $\pi \in \Pi(K)$) is given by $R^2\left(P_j^\pi \cup \{j\}\right) - R^2\left(P_j^\pi\right)$, and the Shapley value can be calculated from

$$Sh_j\left(K, R^2\right) = \frac{1}{k!} \sum_{\pi \in \Pi(K)} R^2\left(P_j^\pi \cup \{j\}\right) - R^2\left(P_j^\pi\right) \ .$$

Note that the Shapley value does not account for a partition. Further, if player j is a Null player, i.e. if $R^2(T \cup \{j\}) - R^2(T) = 0$ for all $T \subseteq K \backslash \{j\}$, we have $Sh_j\left(K, R^2\right) = 0$.

For the actual computation we use the so-called potential function (Hart and Mas-Colell, 1989), defined recursively by $f(T) = R^2(T)/|T| + \sum_{j \in T} f(T \backslash \{j\})/|T|$ and $f(\emptyset) = 0$ since $f(K) - f(K \backslash \{j\}) = Sh_j\left(K, R^2\right)$. Calculation of the Shapley value this way avoids employing $k!$ permutations. For faster computation, we obtain $R^2(T)$ from

the covariance matrix of the data rather than from the original observations (Grömping, 2006, p. 13).

2.2.2 Groups, the Owen value, and the Shapley value

We present two approaches to the Owen value, the first of which applies permutations. We say that a permutation π is compatible with respect to the partition $\mathcal{G}$ if each group's members appear contiguously in the permutation with no player from another group in between, i.e. for $j, l \in G_\ell \in \mathcal{G}$ we have $\pi(j) \leq \pi(k) \leq \pi(l)$ implies $k \in G_\ell$. Let $\Pi(K, \mathcal{G})$ denote the set of all permutations on K that are compatible with $\mathcal{G}$. The Owen value can be obtained from

$$Ow_j\left(K, R^2, \mathcal{G}\right) = \frac{1}{|\Pi(K, \mathcal{G})|} \sum_{\pi \in \Pi(K, \mathcal{G})} R^2\left(P_j^\pi\right) - R^2\left(P_j^\pi \setminus \{j\}\right).$$

In words, variable j's marginal contribution to explanatory value does only count if the groups to which j does not belong to are either completely present or completely absent.

Second, departing from the Shapley value, one may derive the Owen value as follows. If all variables constitute groups of their own such that $\mathcal{G} = \{\{1\}, ..., \{j\}, ..., \{k\}\}$, then each group should be treated as if it were a variable. In general, the distribution of $R^2(K)$ among the groups by using the Shapley value on the external game $\left(\mathcal{G}, R^2_{\mathcal{G}}\right)$ suggests itself (one may ignore the partition whenever applying the Shapley value). It remains to allocate group G_ℓ's payoff $Sh_{G_\ell}\left(\mathcal{G}, R^2_{\mathcal{G}}\right)$ among the variables in G_ℓ.

Consider the case when all variables belong to one group, $G_\ell = K$ and $\mathcal{G} = \{G_\ell\}$. It is desirable to have the same procedure within G_ℓ as if there was no grouping. For that, denote for every subgroup $C \subseteq G_\ell$ the value of that subgroup by $r^2(C)$. Then apply the Shapley value on the resulting internal game $\left(G_\ell, r^2\right)$. If all variables are in G_ℓ, we have $r^2(C) = R^2(C)$ and observe $Sh_j\left(G_\ell, r^2\right) = Sh_j\left(K, R^2\right)$.

Finally, if G_ℓ consists of some but not all variables, a consistent definition of the internal game $\left(G_\ell, r^2\right)$ calls for a reasonable prescription of the worth $r^2(C)$. This can be achieved as follows: Let C participate in the game between groups instead of G_ℓ and determine its Shapley value, i.e. with $\mathcal{G}' = (\mathcal{G} \setminus \{G_\ell\}) \cup \{C\}$ we have $r^2(C) = Sh_C\left(\mathcal{G}', R^2_{\mathcal{G}'}\right)$. Application of the Shapley value on the resulting internal game yields the Owen value, i.e. for $j \in G_\ell$ we have

$$Ow_j\left(K, R^2, \mathcal{G}\right) = Sh_j\left(G_\ell, r^2\right).$$

Note that as a consequence of this setup, the payoffs of the variables of a particular group sum up to the Shapley value from the external game of the respective group, i.e. we have $\sum_{j \in G_\ell} Ow_j\left(K, R^2, \mathcal{G}\right) = Sh_{G_\ell}\left(\mathcal{G}, R^2_{\mathcal{G}}\right)$.

3 Application to German wage data

As an illustration of the method we estimate an augmented Mincer regression model for male workers in Germany. Our focus is on the importance of human capital measures on wages. We use data from the 2006 wave of the German Socio-Economic Panel (GSOEP), which features a symbol-digit correspondence test (SCT) for those participants who took the CAPI interview (Wagner et al., 2007; Lang, 2005).[2] The outcome of this test reflects, to a certain degree, innate cognitive abilities, and these could be regarded as important for an individual's work productivity. To simplify interpretation, we rescale SCT such that it varies between 0 (lowest score) and 1 (highest score). The other human capital variables we consider are years of schooling (EDUC), actual years of work experience (EXPER), and years of employment at the current firm (TENURE).[3] Apart from human capital, we consider whether a man is married (yes/no), the size of the firm (4 categories), industry codes (7 categories), and regions within Germany (15 categories).[4] The dependent variable is the natural logarithm of hourly earnings. We restrict the sample to male German citizens, aged 20–64 years, who worked for at least 10 hours per week, were not self-employed, and not disabled. This leaves us with 850 observations with valid data.

The overall R^2 of our model is 0.497 (Table 1). More than half (55.9%) of this result—or 28% of the entire variation in log wages—can be attributed to the human capital variables in the model, whereas the second-largest group, regional disparity, accounts for only one sixth of the explained variance. Further decomposition of the R^2 share of human capital reveals that the main effect of EDUC is the single most "important" variable, with a 22% share of the overall R^2. A similar magnitude is achieved by the labor market and firm experience variables, though, if we sum up their respective Owen value shares (22.8%). Despite its "large" coefficient and statistical significance, the main effect of cognitive ability does not turn out to be very important in terms of explanatory power. Its interaction term with EDUC, however, contributes 8.1% of the model's R^2. The coefficients imply that up to about 16 years of schooling, greater cognitive abilities result in higher earnings.[5]

Calculating confidence intervals for our results is a computationally tedious task, since we rely on the bootstrap method. Figure 1 presents confidence intervals for the absolute group or Owen values (without standardizing by R^2), on the basis of 2000

[2] See Anger and Heineck (2010) for a more detailed analysis of wages and SCT based on the GSOEP and Isreali (2007) for a decomposition analysis for a wage regression model.

[3] EDUC is measured as years of schooling that would be necessary to achieve the person's highest school or college degree, irrespective of how long it actually took.

[4] The regions are equivalent to German *Laender* (NUTS 1) with the exception of Rhineland-Palatinate and Saarland, which are one region in the GSOEP.

[5] Measurement error in our indicator of cognitive ability could be responsible for relatively low predictive power. Suffice to say, measurement error is a general problem in empirical work, not only when decomposing goodness of fit.

Table 1: Wage regression results

Group / regressor	Coef.	R^2 decomposition (%) Group	Owen
Human capital	(6 variables)**	55.9	
SCT	0.783*		3.0
SCT × EDUC	-0.050*		8.1
EDUC	0.104**		22.0
EXPER	0.031**		8.1
$(\text{EXPER})^2/100$	-0.053**		4.9
TENURE	0.008**		9.8
Married	0.083**	6.1	
Firm size	(3 dummies)**	15.7	
Industry	(6 dummies)**	5.8	
Region	(14 dummies)**	16.6	
Intercept	0.527*		
Observations	850		
Overall R^2	0.497		

Remarks: */** denotes statistical significance at the 10% / 5% level for individual variables (t-test) or groups of dummy variables (F-test), based on the heteroscedasticity-robust covariance matrix.

bootstrap replications.[6] With 90% confidence we find that between one quarter and about one third of the entire variation in log wages can be attributed to human capital.

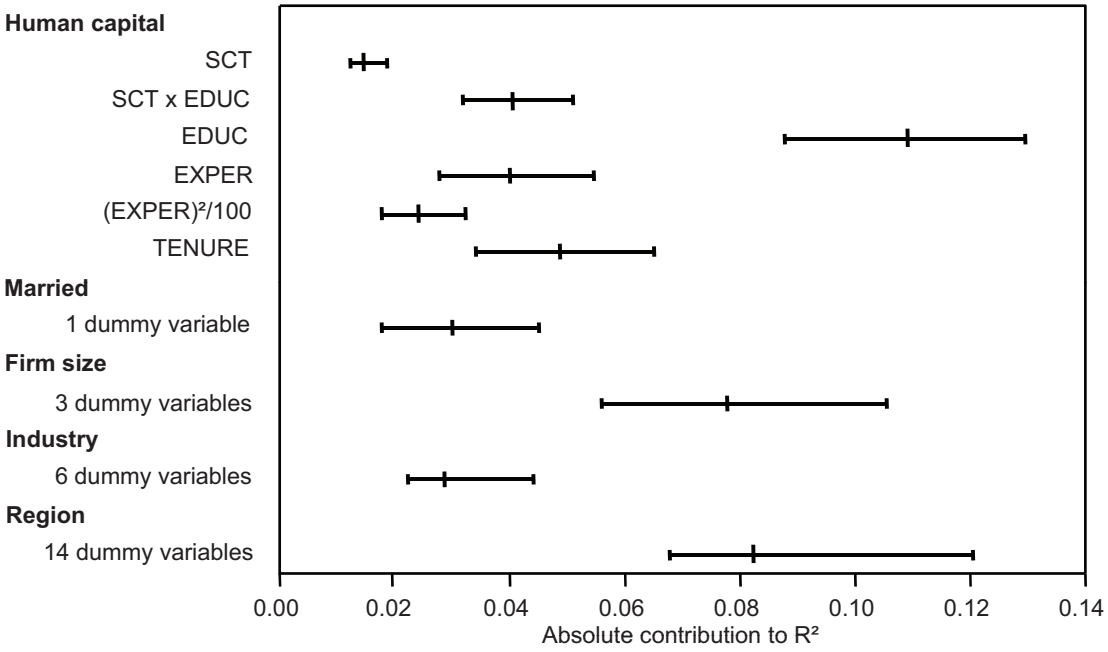
4 Concluding remarks

Decomposition of goodness-of-fit can be an attractive diagnostic tool for the identification of important (groups of) explanatory variables in a given model. Unlike statistical significance, R^2 is not a function of the number of observations. Another practical application could consist in the judgment of the value of control variables in the context of survey design, if preliminary data already exist and if data collection is costly.

Both the Shapley and the Owen value reflect average marginal contribution to R^2. As such, a non-zero value may be assigned even to a variable that exhibits a zero coefficient in the full model. Compared to calculation of the Shapley value for each variable, decomposition based on the Owen value comes at lower computational cost if prior knowledge on the grouping of regressors can be exploited.

[6] Using the software Stata 11.2 (StataCorp, 2009), this bootstrap exercise required about 7½ minutes on a desktop PC, whereas estimation of the values in Table 1 only took ¼ second. A user-written program and a syntax file to replicate the results is available upon request from the authors.

Figure 1: Decomposition results with 90% bootstrap confidence intervals



Remarks: Based on 2000 bootstrap replications. Confidence interval for "human capital" group: [0.246; 0.313].

While the present procedure is confined to the case of linear models with independent observations and simple partitions, several extensions can be conceived. First, groups could be further disaggregated, giving rise to level structures (Winter, 1989). For instance, labor market experience and its squared term could form an intermediate group in our application. Second, the model could allow for unit-specific fixed effects that control for time-invariant unobserved heterogeneity, so that only the "within-R^2" is decomposed. Third, the method could be applied to non-linear models estimated by the maximum likelihood method, admittedly at a much higher computational burden. In this case the likelihood ratio test statistic could be allocated across explanatory entities.

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